Hamiltonian structures of fermionic two-dimensional Toda lattice hierarchies [1]

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Abstract

By exhibiting the corresponding Lax pair representations we propose a wide class of integrable two-dimensional (2D) fermionic Toda lattice (TL) hierarchies which includes the 2D $N = (2|2)$ and $N = (0|2)$ supersymmetric TL hierarchies as particular cases. We develop the generalized graded R-matrix formalism using the generalized graded bracket on the space of graded operators with involution generalizing the graded commutator in superalgebras, which allows one to describe these hierarchies in the framework of the Hamiltonian formalism and construct their first two Hamiltonian structures. The first Hamiltonian structure is obtained for both bosonic and fermionic Lax operators while the second Hamiltonian structure is established for bosonic Lax operators only.

1 Introduction

The 2D TL hierarchy was first studied in [1, 2], and at present two different nontrivial supersymmetric extensions of 2D TL are known. They are the $N = (2|2)$ [3]-[12] and $N = (0|2)$ [12, 9] supersymmetric TL hierarchies that possess a different number of supersymmetries and contain the $N = (2|2)$ and $N = (0|2)$ TL equations as subsystems. Quite recently, the 2D generalized fermionic TL equations have been introduced [11] and their two reductions related to the $N = (2|2)$ and $N = (0|2)$ supersymmetric TL equations were considered. In the present paper, we describe a wide class of integrable two-dimensional fermionic Toda lattice hierarchies which includes the 2D $N = (2|2)$ and $N = (0|2)$ supersymmetric TL hierarchies as particular cases and contains the 2D generalized fermionic TL equations as a subsystem.

The Hamiltonian description of the 2D TL hierarchy has been constructed only quite recently in the framework of the R-matrix approach in [13], where the new R-matrix associated with splitting of algebra given by a pair of difference operators was introduced. In the present

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paper, we adapt this R-matrix to the case of Z_2-graded operators and derive the bi-Hamiltonian structure of the 2D fermionic (K, M)-TL hierarchy.

Remarkably, in solving this problem the generalized graded bracket (7) on the space of graded operators with an involution finds its new application. This bracket was introduced in [14], where it was observed that the $N = (1|1)$ supersymmetric 2D TL hierarchy had a natural Lax-pair representation in terms of this bracket which allowed one to derive the dispersionless $N = (1|1)$ 2D TL hierarchy and its Lax representation. In the present paper, the generalized graded bracket is used to describe the 2D fermionic (K, M)-TL hierarchy and define its two Hamiltonian structures.

The structure of this paper is as follows. In Sec. 2, we define the space of the Z_2-graded difference operators with the involution and recall the generalized graded bracket [14] and its properties. In Sec. 3, we give a theoretical background of the R-matrix method generalized to the case of the Z_2-graded difference operators. We define the R-matrix on the associative algebra g of the Z_2-graded difference operators, derive the graded modified Yang-Baxter equation and using the generalized graded bracket obtain two Poisson brackets for the functionals on $\mathsf{g}^{\dagger} = \mathsf{g}$. Using these Poisson brackets one can define the Hamiltonian equations that can equivalently be rewritten in terms of the Lax-pair representation. The basic results of Sec. 3 are formulated as Theorem. In Sec. 4, using the generalized graded bracket we propose a new 2D fermionic (K, M)-TL hierarchy in terms of the Lax-pair representation and construct the algebra of its flows. In Sec. 5, we apply the results of Sec. 3 to derive the Hamiltonian structures of the 2D fermionic (K, M)-TL hierarchy. In Sec. 6, we briefly summarize the main results obtained in this paper and point out open problems.

2 Space of difference operators

In this section we define the space of difference operators which will play an important role in our consideration. These operators can be represented in the following general form:

$$\mathbb{O}_m = \sum_{k=-\infty}^{\infty} f_{k,j}^{(m)} \, e^{(k-m)\partial}, \qquad m, j \in \mathbb{Z}, \tag{1}$$

parameterized by the functions $f_{2k,j}^{(m)}$ ($f_{2k+1,j}^{(m)}$) which are the Z_2-graded bosonic (fermionic) lattice fields with the lattice index j ($j \in \mathbb{Z}$) and the Grassmann parity defined by index k

$$d_{f_{k,j}^{(m)}} = |k| \bmod 2.$$

In what follows we suppose that the functions $f_{k,j}^{(m)}$ obey the zero boundary conditions at infinity

$$\lim_{j \to \pm\infty} f_{k,j}^{(m)} = 0. \tag{2}$$

$e^{k\partial}$ is the shift operator whose action on the lattice fields results into a discrete shift of a lattice index

$$e^{l\partial} f_{k,j}^{(m)} = f_{k,j+l}^{(m)} e^{l\partial}. \tag{3}$$

The shift operator has Z_2-parity defined as

$$d'_{e^{l\partial}} = |l| \bmod 2.$$

The operators $\mathbb{O}_m$ (1) admit the diagonal Z_2-parity

$$d_{\mathbb{O}_m} = d_{f^{(m)}_{k,j}} + d'_{e^{(k-m)\partial}} = |m| \bmod 2 \tag{4}$$

and the involution

$$\mathbb{O}^*_m = \sum_{k=-\infty}^{\infty} (-1)^k f^{(m)}_{k,j} \ e^{(k-m)\partial}.$$

In what follows we also need the projections of the operators $\mathbb{O}_m$ defined as

$$(\mathbb{O}_m)_{\leqslant p} = \sum_{k \leqslant p+m} f_{k,j} \ e^{(k-m)\partial}, \qquad (\mathbb{O}_m)_{\geqslant p} = \sum_{k \geqslant p+m} f_{k,j} \ e^{(k-m)\partial}$$

and we will use the usual notation for the projections $(\mathbb{O}_m)_+ := (\mathbb{O}_m)_{\geqslant 0}$ and $(\mathbb{O}_m)_- := (\mathbb{O}_m)_{<0}$. Note that $e^{l\partial}$ is a conventional form for the shift operators defined in terms of infinite-dimensional matrices $(e^{l\partial})_{i,j} \equiv \delta_{i,j-l}$, and there is an isomorphism between operators (1) and infinite-dimensional matrices (see e.g. [15])

$$\mathbb{O}_m = \sum_{k=-\infty}^{\infty} f^{(m)}_{k,j} \ e^{(k-m)\partial} \quad \rightarrow \quad (\mathbb{O}_m)_{j,i} \equiv \sum_{k=-\infty}^{\infty} f^{(m)}_{k,j} \ \delta_{j,i-k+m}.$$

In the operator space (1) one can extract two subspaces which are of great importance in our further consideration

$$\mathbb{O}^+_{K_1} = \sum_{k=0}^{\infty} f_{k,j} \ e^{(K_1-k)\partial}, \qquad K_1 \in \mathbb{N}, \tag{5}$$

$$\mathbb{O}^-_{K_2} = \sum_{k=0}^{\infty} f_{k,j} \ e^{(k-K_2)\partial}, \qquad K_2 \in \mathbb{N}. \tag{6}$$

The operators of the subspaces $\mathbb{O}^+_{K_1}$ and $\mathbb{O}^-_{K_2}$ form associative algebras with the multiplication (3). Using this fact we define on these subspaces the generalized graded algebra with the bracket [14]

$$[\mathbb{O}, \widetilde{\mathbb{O}}\} := \mathbb{O} \ \widetilde{\mathbb{O}} - (-1)^{d_{\mathbb{O}} d_{\widetilde{\mathbb{O}}}} \ \widetilde{\mathbb{O}}^{*(d_{\mathbb{O}})} \ \mathbb{O}^{*(d_{\widetilde{\mathbb{O}}})}, \tag{7}$$

where the operators $\mathbb{O}$ and $\widetilde{\mathbb{O}}$ belong to the subspaces $\mathbb{O}^+_{K_1}$ $(\mathbb{O}^-_{K_2})$, and $\mathbb{O}^{*(m)}$ denotes the m-fold action of the involution $*$ on the operator $\mathbb{O}$, $(\mathbb{O}^{*(2)} = \mathbb{O})$. Bracket (7) generalizes the (anti)commutator in superalgebras and satisfies the following properties [14]:

symmetry

$$[\mathbb{O}, \widetilde{\mathbb{O}}\} = -(-1)^{d_{\mathbb{O}} d_{\widetilde{\mathbb{O}}}} \ [\widetilde{\mathbb{O}}^{*(d_{\mathbb{O}})}, \mathbb{O}^{*(d_{\widetilde{\mathbb{O}}})}\}, \tag{8}$$

derivation

$$[\mathbb{O}, \widetilde{\mathbb{O}}\,\widehat{\mathbb{O}}\} = [\mathbb{O}, \widetilde{\mathbb{O}}\}\,\widehat{\mathbb{O}} + (-1)^{d_{\mathbb{O}} d_{\widetilde{\mathbb{O}}}}\,\widetilde{\mathbb{O}}^{*(d_{\mathbb{O}})}[\mathbb{O}^{*(d_{\widetilde{\mathbb{O}}})}, \widehat{\mathbb{O}}\}, \tag{9}$$

and Jacobi identity

$$\begin{aligned}(-1)^{d_{\mathbb{O}} d_{\widehat{\mathbb{O}}}}\,[[\mathbb{O},\, \widetilde{\mathbb{O}}^{*(d_{\mathbb{O}})}\},\, \widehat{\mathbb{O}}^{*(d_{\mathbb{O}}+d_{\widetilde{\mathbb{O}}})}\} + (-1)^{d_{\widetilde{\mathbb{O}}} d_{\mathbb{O}}}\,[[\widetilde{\mathbb{O}},\, \widehat{\mathbb{O}}^{*(d_{\widetilde{\mathbb{O}}})}\},\, \mathbb{O}^{*(d_{\widetilde{\mathbb{O}}}+d_{\widehat{\mathbb{O}}})}\}& \\ +\ (-1)^{d_{\widehat{\mathbb{O}}} d_{\widetilde{\mathbb{O}}}}\,[[\widehat{\mathbb{O}},\, \mathbb{O}^{*(d_{\widehat{\mathbb{O}}})}\},\, \widetilde{\mathbb{O}}^{*(d_{\widehat{\mathbb{O}}}+d_{\mathbb{O}})}\} \ &= \ 0. \end{aligned} \tag{10}$$

For the operators $\mathbb{O}_m$ (1) we define the supertrace

$$str\mathbb{O} = \sum_{j=-\infty}^{\infty} (-1)^j f_{m,j}^{(m)}. \tag{11}$$

One can easily verify that the main property of supertraces $str[\mathbb{O}, \widetilde{\mathbb{O}}\} = 0$ is indeed satisfied for the case of the generalized graded bracket (7) if the functions entering into operators (1) obey the zero boundary condition at infinity (2).

3 R-matrix formalism

In this section, we develop a theoretical background of the R-matrix method adapted to the case of the operator space (1).

Let g be an associative algebra of the operators from the space (1) with the invariant non-degenerate inner product

$$< \mathbb{O}, \widetilde{\mathbb{O}} >= str(\mathbb{O}\,\widetilde{\mathbb{O}})$$

using which one can identify the algebra g with its dual $\mathsf{g}^{\dagger}$. We set the following Poisson bracket:

$$\{f, g\}(\mathbb{O}) = - < \mathbb{O}, [\nabla g, (\nabla f)^{*(\nabla d_g)}\} >, \tag{12}$$

where f, g are functionals on g, and ∇f and ∇g are their gradients at the point $\mathbb{O}$ which are related with f, g through the inner product

$$\left.\frac{\partial f(\mathbb{O} + \epsilon\delta\mathbb{O})}{\partial\epsilon}\right|_{\epsilon=0} = < \delta\mathbb{O}, \nabla f(\mathbb{O}) > .$$

Note that the proper properties of the Poisson bracket (12) follow from the properties (8–10) of the generalized bracket (7) and are strictly determined by the Z_2-parity of the operator $\mathbb{O}$. Thus, one has symmetry

$$\{f, g\} \ = \ -(-1)^{(d_f+d_{\mathbb{O}})(d_g+d_{\mathbb{O}})}\{g, f\}, \tag{13}$$

derivation

$$\{f, gh\} \ = \ \{f, g\}h + (-1)^{d_g(d_f+d_{\mathbb{O}})} g\{f, h\}, \tag{14}$$

and Jacobi identity

$$(-1)^{(d_f+d_{\mathbb{O}})(d_h+d_{\mathbb{O}})}\{\{f,g\},h\}+(-1)^{(d_g+d_{\mathbb{O}})(d_f+d_{\mathbb{O}})}\{\{g,h\},f\}$$
$$+(-1)^{(d_h+d_{\mathbb{O}})(d_g+d_{\mathbb{O}})}\{\{h,f\},g\} \;=\; 0. \qquad (15)$$

Therefore, for the even operator $\mathbb{O}$ one has usual (even) Z_2-graded Poisson bracket, while for the operators with odd diagonal parity $d_{\mathbb{O}}$ (4) eq. (12) defines odd Z_2-graded Poisson bracket (antibracket).

Having defined the Poisson bracket we proceed with the search for the hierarchy of flows generated by this bracket using Hamiltonians. Therefore, we need to determine an infinite set of functionals which should be in involution to play the role of Hamiltonians. For Poisson bracket (12) one can find an infinite set of Hamiltonians in a rather standard way

$$H_k=\frac{1}{k}str\mathbb{O}_*^k=\frac{1}{k}\sum_{i=-\infty}^{\infty}(-1)^i f_{km,i}^{(km)}, \qquad (16)$$

where $\mathbb{O}_*^k$ is defined as

$$(\mathbb{O})_*^{2k}:=(\mathbb{O}^{*(d_{\mathbb{O}})}\mathbb{O})^k, \quad (\mathbb{O})_*^{2k+1}:=\mathbb{O}\ (\mathbb{O})_*^{2k}. \qquad (17)$$

For the odd operators $\mathbb{O}$ eq. (16) defines only fermionic nonzero functionals H_{2k+1}, since in this case even powers of the operators $\mathbb{O}$ have the following representation:

$$d_{\mathbb{O}}=1: \quad (\mathbb{O})_*^{2k}=(1/2[(\mathbb{O})^*,\mathbb{O}\})^k\equiv 1/2[((\mathbb{O})_*^{2k-1})^*,\mathbb{O}\} \qquad (18)$$

and all the bosonic Hamiltonians are trivial ($H_{2k}=0$) like the supertrace of the generalized graded bracket.

The functionals (16) are obviously in involution but produce a trivial dynamics. Actually, the functionals H_k (16) are the Casimir operators of the Poisson bracket (12), so the Poisson bracket of H_k with any other functional is equal to zero as an output (due to the relation $\nabla H_{k+1}=\mathbb{O}_*^k$). Nevertheless, it is possible to modify the Poisson bracket (12) in such a way that the new Poisson bracket would produce nontrivial equations of motion using the same Hamiltonians (16) and these Hamiltonians are in involution with respect to the modified Poisson bracket as well. Let us introduce the modified generalized graded bracket on the space (1)

$$[\mathbb{O},\widetilde{\mathbb{O}}\}_R:=[R(\mathbb{O}),\widetilde{\mathbb{O}}\}+[\mathbb{O},R(\widetilde{\mathbb{O}})\}, \qquad (19)$$

where the R-matrix is a linear map R: $\mathsf{g}\rightarrow\mathsf{g}$ such that the bracket (19) satisfies the properties (8–10). One can verify that the Jacobi identities (10) for the bracket (19) can equivalently be rewritten in terms of the generalized graded bracket (7)

$$(-1)^{d_{\mathbb{O}}d_{\widehat{\mathbb{O}}}}\,[[\mathbb{O},\,\widetilde{\mathbb{O}}^{*(d_{\mathbb{O}})}\}_R,\,\widehat{\mathbb{O}}^{*(d_{\mathbb{O}}+d_{\widetilde{\mathbb{O}}})}\}_R+\text{cycle perm.}=$$
$$(-1)^{d_{\mathbb{O}}d_{\widehat{\mathbb{O}}}}\,[R([\mathbb{O},\,\widetilde{\mathbb{O}}^{*(d_{\mathbb{O}})}\}_R)-[R(\mathbb{O}),\,R(\widetilde{\mathbb{O}}^{*(d_{\mathbb{O}})})\},\widehat{\mathbb{O}}^{*(d_{\mathbb{O}}+d_{\widetilde{\mathbb{O}}})}\}+\text{cycle perm.}=0.$$

Thus, one can conclude that a sufficient condition for R to be the R-matrix is the validity of the following equation:

$$R([\mathbb{O},\widetilde{\mathbb{O}}\}_R)-[R(\mathbb{O}),R(\widetilde{\mathbb{O}})\}=\alpha[\mathbb{O},\widetilde{\mathbb{O}}\}, \qquad (20)$$

where α is an arbitrary constant. Following the terminology of [16] we call eq. (20) the graded modified Yang-Baxter equation. Eq. (20) is the generalization of the graded modified classical Yang-Baxter equation discussed in paper [17] for the space of graded operators (1).

With the new bracket (19) one can define the corresponding new Poisson bracket on dual $\mathsf{g}^\dagger$

$$\{f,g\}_1(\mathbb{O}) \quad = \quad -1/2 < \mathbb{O}, [\nabla g, (\nabla f)^{*(d_{\nabla g})}\}_R > . \qquad (21)$$

With respect to the dependence of the r.h.s of (21) on the point $\mathbb{O}$ this is a linear bracket. Without going into details we introduce also bi-linear bracket for bosonic graded operators $\mathbb{O}_B$ ($d_{\mathbb{O}_B} = 0$) as follows:

$$\begin{aligned} \{f,g\}_2(\mathbb{O}_B) \quad &= \quad -1/4 < [\mathbb{O}_B, \nabla g\} R((\nabla f)^{*(d_{\nabla g})} \mathbb{O}_B^{*(d_{\nabla f}+d_{\nabla g})} \\ &+ \quad \mathbb{O}_B^{*(d_{\nabla g})} (\nabla f)^{*(d_{\nabla g})}) - R(\nabla g \mathbb{O}_B^{*(d_{\nabla g})} + \mathbb{O}_B \nabla g)[\mathbb{O}_B^{*(d_{\nabla g})}, (\nabla f)^{*(d_{\nabla g})}\} > . \qquad (22) \end{aligned}$$

We did not succeed in constructing the bi-linear bracket for the case of fermionic operators $\mathbb{O}_F$ ($d_{\mathbb{O}_F} = 1$). The bracket (21) is obviously the Poisson bracket if R is an R-matrix on g. The bi-linear bracket (22) becomes Poisson bracket under more rigorous constraints which can be found in the following

Theorem. a) Linear bracket (21) is the Poisson bracket if R obeys the graded modified Yang-Baxter equation (20);
b) the bi-linear bracket (22) is the Poisson bracket if R and its skew-symmetric part $1/2(R - R^\dagger)$ obey the graded modified Yang-Baxter equation (20) with the same α, where the adjoint operator $R^\dagger$ acts on the dual $\mathsf{g}^\dagger$

$$< \mathbb{O}, R(\widetilde{\mathbb{O}}) >=< R^\dagger(\mathbb{O}), \widetilde{\mathbb{O}} >;$$

c) if $\mathbb{O} = \mathbb{O}_B$, then these two Poisson brackets are compatible

$$\{f,g\}_2(\mathbb{O}_B + b) = \{f,g\}_2(\mathbb{O}_B) + b\{f,g\}_1(\mathbb{O}_B);$$

d) the Casimir operators H_k (16) of the bracket (12) are in involution with respect to both linear (21) and bi-linear (22) Poisson brackets;
e) the Hamiltonians $H_k \neq 0$ (16) generate evolution equations

$$\begin{aligned} \partial_k \mathbb{O} \quad &= \quad \{H_{k+1}, \mathbb{O}\}_1 = 1/2[R((\nabla H_{k+1})^{*(d_{\mathbb{O}})}), \mathbb{O}\}, \\ \partial_k \mathbb{O}_B \quad &= \quad \{H_k, \mathbb{O}_B\}_2 = 1/4[R(\nabla H_k \mathbb{O}_B + \mathbb{O}_B \nabla H_k), \mathbb{O}_B\} \end{aligned}$$

via the brackets (21) and (22), respectively, which connect the Lax-pair and Hamiltonian representations. ▼

Note that in the case when the shift operators and functions parameterizing the difference operators $\mathbb{O}$ (1) have even Z_2-parity the similar Theorem was discussed in [16, 18, 19, 13].

4 2D fermionic (K, M)-Toda lattice hierarchy

In this section, we introduce the two-dimensional fermionic (K, M)-Toda lattice hierarchy in terms of the Lax-pair representation.

Let us consider two difference operators L_K^+ and L_M^-

$$L_K^+ = \sum_{k=0}^{\infty} u_{k,i} e^{(K-k)\partial}, \qquad L_M^- = \sum_{k=0}^{\infty} v_{k,i} e^{(k-M)\partial}, \tag{23}$$

which obviously belong to the spaces (5) and (6), respectively. The lattice fields and the shift operator entering into these operators have the following length dimensions: $[u_{k,i}] = -\frac{1}{2}k$, $[v_{k,i}] = \frac{1}{2}(k-K-M)$ and $[e^{k\partial}] = -\frac{1}{2}k$, respectively, so operators (23) are of equal length dimension, $[L_K^+] = [L_M^-] = -\frac{1}{2}K$. The dynamics of the fields $u_{k,i}, v_{k,i}$ are governed by the Lax equations expressed in terms of the generalized bracket (7) [14]

$$\begin{aligned} D_s^{\pm} L_{\Omega^\alpha}^{\alpha} &= \mp\alpha(-1)^{s\Omega^\alpha\Omega^\pm}[(((L_{\Omega^\pm}^{\pm})_*^s)_{-\alpha})^{*(\Omega^\alpha)}, L_{\Omega^\alpha}^{\alpha}\}, \\ & \qquad \alpha = +,-, \quad \Omega^+ = K, \quad \Omega^- = M, \quad s \in \mathbb{N}, \end{aligned} \tag{24}$$

where $D_s^{\pm}$ are evolution derivatives with the Z_2-parity defined as

$$d_{D_s^+} = sK \mod 2, \qquad d_{D_s^-} = sM \mod 2$$

and the length dimension $[D_s^+] = [D_s^-] = -sK/2$. The Lax equations (24) generate non-Abelian (super)algebra of flows of the 2D fermionic (K,M)-TL hierarchy

$$[D_s^+, D_p^+\} = (1-(-1)^{spK})D_{s+p}^+, \quad [D_s^-, D_p^-\} = (1-(-1)^{spM})D_{s+p}^-, \quad [D_s^+, D_p^-\} = 0.$$

The composite operators $(L_K^+)_*^s$ and $(L_M^-)_*^s$ entering into the Lax equations (24) are defined by eq. (17) and also belong to the spaces (5) and (6), respectively,

$$(L_K^+)_*^r := \sum_{k=0}^{\infty} u_{k,i}^{(r)} e^{(rK-k)\partial}, \qquad (L_M^-)_*^r := \sum_{k=0}^{\infty} v_{k,i}^{(r)} e^{(k-rM)\partial}.$$

Here $u_{k,i}^{(r)}$ and $v_{k,i}^{(r)}$ are functionals of the original fields and there are the following recursion relations for them

$$\begin{aligned} u_{p,i}^{(r+1)} &= \sum_{k=0}^{p} (-1)^{kK} u_{k,i}^{(r)} u_{p-k,i-k+rK}, \quad u_{p,i}^{(1)} = u_{p,i}, \\ v_{p,i}^{(r+1)} &= \sum_{k=0}^{p} (-1)^{kM} v_{k,i}^{(r)} v_{p-k,i+k-rM}, \quad v_{p,i}^{(1)} = v_{p,i}. \end{aligned}$$

Now using the Lax representation (24) and relations (9) and (17) one can derive the equations of motion for the composite Lax operators

$$D_s^{\pm}(L_{\Omega^\alpha}^{\alpha})_*^r = \mp\alpha(-1)^{sr\Omega^\alpha\Omega^\pm}[(((L_{\Omega^\pm}^{\pm})_*^s)_{-\alpha})^{*(r\Omega^\alpha)}, (L_{\Omega^\alpha}^{\alpha})_*^r\}. \tag{25}$$

The Lax-pair representation (25) generates the following equations for the functionals $u_{k,i}^{(r)}, v_{k,i}^{(r)}$:

$$D_s^+ u_{k,i}^{(r)} = \sum_{p=1}^{k}((-1)^{rpK+1} u_{p+sK,i}^{(s)} u_{k-p,i-p}^{(r)} + (-1)^{(k+p)sK} u_{k-p,i}^{(r)} u_{p+sK,i+p-k+rK}^{(s)}),$$

$$
\begin{aligned}
D_s^- u_{k,i}^{(r)} &= \sum_{p=0}^{sM-1}((-1)^{(sM+p)rK} v_{p,i}^{(s)} u_{p+k-sM,i+p-sM}^{(r)} - (-1)^{(k+p+1)sM} u_{p+k-sM,i}^{(r)} v_{p,i-p-k+sM+rK}^{(s)}),\\
D_s^+ v_{k,i}^{(r)} &= \sum_{p=0}^{sK}((-1)^{(sK+p)rM} u_{p,i}^{(s)} v_{p+k-sK,i-p+sK}^{(r)} - (-1)^{(k+p+1)sK} v_{p+k-sK,i}^{(r)} u_{p,i+p+k-sK-rM}^{(s)}),\\
D_s^- v_{k,i}^{(r)} &= \sum_{p=0}^{k}((-1)^{rpM+1} v_{p+sM,i}^{(s)} v_{k-p,i+p}^{(r)} + (-1)^{(k+p)sM} v_{k-p,i}^{(r)} v_{p+sM,i+k-p-rM}^{(s)}). \qquad (26)
\end{aligned}
$$

It is assumed that in the right-hand side of eqs. (26) all the functionals $u_{k,i}^{(r)}$ $v_{k,i}^{(r)}$ with $k < 0$ should be set equal to zero.

One can demonstrate that all known up to now fermionic 2D Toda lattice equations [6]-[12] can be reproduced from the system of equations (26) as subsystems with additional reduction constraints imposed. We call equations (24) the 2D fermionic (K,M)-Toda lattice hierarchy.

5 Bi-Hamiltonian structure of 2D fermionic (K, M)-TL hierarchy

In this section, we apply the R-matrix approach to build the bi-Hamiltonian structure of the 2D fermionic (K,M)-TL hierarchy. This hierarchy is associated with two Lax operators (23) belonging to the operator space (5-6). Following [13] we consider the associative algebra on the space of the direct sum of two difference operators

$$
\mathsf{g} := \mathbb{O}_{K_1}^+ \oplus \mathbb{O}_{K_2}^-. \qquad (27)
$$

However, in contrast to the case of pure bosonic 2D TL hierarchy, the difference operators in the direct sum (27) can be of both opposite and equal diagonal Z_2-parity. It turns out that the Poisson bracket can correctly be defined only for the latter case. In what follows we restrict ourselves to the case when both operators in g (27) have the same diagonal parity.

We denote (x^+, x^-) elements of such algebra $\mathsf{g} = \mathsf{g}^\dagger$ with the product

$$
(x_1^+, x_1^-) \cdot (x_2^+, x_2^-) = (x_1^+ x_2^+, x_1^- x_2^-), \qquad (28)
$$

and define the inner product as follows:

$$
< (x^+, x^-) > := str(x^+ + x^-), \qquad (29)
$$

where $x^+ \in \mathbb{O}_{K_1}^+$, $x^- \in \mathbb{O}_{K_2}^-$. Using this definition we set the Poisson brackets to be

$$
\{f_1, f_2\} = < (\mathbb{O}_{K_1}^+, \mathbb{O}_{K_2}^-), [\nabla f_1, \nabla f_2\}^{\oplus} >, \qquad (30)
$$

where

$$
[\nabla f_1, \nabla f_2\}^{\oplus} := ([\nabla f_1^+, (\nabla f_2^+)^{*(d_{\nabla f_1^+})}\}, [\nabla f_1^-, (\nabla f_2^-)^{*(d_{\nabla f_1^-})}\}),
$$

f_k are functionals on g (27), and $\nabla f_k[(\mathbb{O}^+_{K_1},\mathbb{O}^-_{K_2})]=(\nabla f^+_k,\nabla f^-_k)$ are their gradients which can be found from the definition

$$\left.\frac{\partial f_k[(\mathbb{O}^+_{K_1},\mathbb{O}^-_{K_2})+\epsilon(\delta\mathbb{O}^+_{K_1},\delta\mathbb{O}^-_{K_2})]}{\partial\epsilon}\right|_{\epsilon=0} = \;<(\delta\mathbb{O}^+_{K_1},\delta\mathbb{O}^-_{K_2}),(\nabla f^+_k,\nabla f^-_k)> \\ = \;<\delta\mathbb{O}^+_{K_1},\nabla f^+_k>+<\delta\mathbb{O}^-_{K_2},\nabla f^-_k>.$$

In order to obtain nontrivial Hamiltonian dynamics, one needs to modify the brackets (30) applying the R-matrix

$$[\nabla f_1,\nabla f_2\}^{\oplus}\quad\longrightarrow\quad[\nabla f_1,\nabla f_2\}^{\oplus}_R=[R(\nabla f_1),\nabla f_2\}^{\oplus}+[\nabla f_1,R(\nabla f_2)\}^{\oplus}.$$

The R-matrix acts on the space (27) in the nontrivial way and mixes up the elements from two subalgebras in the direct sum with each other

$$R(x^+,x^-)=(x^+_+-x^+_-+2x^-_-,x^-_--x^-_++2x^+_+) \tag{31}$$

which is a crucial point of the R-matrix approach in the two-dimensional case [13]. This R-matrix satisfies the graded modified Yang-Baxter equation

$$R([(x^+,x^-),(y^+,y^-)\}_R)-[R(x^+,x^-),R(y^+,y^-)\}=\alpha[(x^+,x^-),(y^+,y^-)\} \tag{32}$$

with $\alpha=1$ and allows one to find two compatible Poisson structures and rewrite Lax-pair representation (24) in the Hamiltonian form. The direct verification by substitution in (32) shows that the skew-symmetric part

$$1/2(R(x^+,x^-)-R^{\dagger}(x^+,x^-))=(x^+_{>0}-x^+_{<0}-x^-_0,x^-_{<0}-x^-_{>0}+x^+_0)$$

also satisfies the graded modified Yang-Baxter equation (32). Therefore, by Theorem of section 3 there exist two Poisson structures on g (27).

Using eqs. (21–22), (28), (31) and cyclic permutations inside the supertrace (11) we obtain the following general form of the first and second Poisson brackets:

$$\{f,g\}_i=<P^+_i(\nabla g^+,\nabla g^-),(\nabla f^+)^{*(d_{\nabla g})}>+<P^-_i(\nabla g^+,\nabla g^-),(\nabla f^-)^{*(d_{\nabla g})}>, \tag{33}$$

where $i=1,2$ and $d_{\nabla g}:=d_{\nabla g^+}=d_{\nabla g^-}$. The Poisson tensors in eq. (33) are found for any values of (K,M) for the first Hamiltonian structure

$$P^+_1(\nabla g^+,\nabla g^-)\;=\;[(\nabla g^-_- -\nabla g^+_-)^{*(K)},(L^+_K)^{*(d_{\nabla g})}\}-([L^+_K,\nabla g^+\}+[L^-_M,\nabla g^-\})_{\leqslant 0},$$
$$P^-_1(\nabla g^+,\nabla g^-)\;=\;[(\nabla g^+_+ -\nabla g^-_+)^{*(M)},(L^-_M)^{*(d_{\nabla g})}\}-([L^+_K,\nabla g^+\}+[L^-_M,\nabla g^-\})_{>0},$$

while for the second Hamiltonian structure we constructed the explicit expression of the Poisson tensors for even values of (K,M) only

$$\begin{aligned}P^+_2(\nabla g^+,\nabla g^-)&=1/2\Big([(\nabla g^-(L^-_M)^{*(d_g)}+L^-_M\nabla g^- -\nabla g^+(L^+_K)^{*(d_g)}-L^+_K\nabla g^+)_-,(L^+_K)^{*(d_g)}\}\\&\quad-L^+_K\;([L^+_K,\nabla g^+\}+[L^-_M,\nabla g^-\})_{\leqslant 0}-([L^+_K,\nabla g^+\}+[L^-_M,\nabla g^-\})_{\leqslant 0}\;(L^+_K)^{*(d_g)}\Big),\\P^-_2(\nabla g^+,\nabla g^-)&=1/2\Big([(\nabla g^+(L^+_K)^{*(d_g)}+L^+_K\nabla g^+ -\nabla g^-(L^-_M)^{*(d_g)}-L^-_M\nabla g^-)_+,(L^-_M)^{*(d_g)}\}\\&\quad-L^-_M\;([L^+_K,\nabla g^+\}+[L^-_M,\nabla g^-\})_{>0}-([L^+_K,\nabla g^+\}+[L^-_M,\nabla g^-\})_{>0}\;(L^-_M)^{*(d_g)}\Big).\end{aligned}$$

The Poisson brackets for the functions $u_{n,i}$ and $v_{n,i}$ parameterizing the Lax operators (23) can explicitly be derived from (33) if one takes into account that

$$\begin{aligned}\nabla u_{n,\xi} &\equiv (\nabla u^{+}_{n,\xi}, \nabla u^{-}_{n,\xi}) = (e^{(n-K)\partial}(-1)^{i}\delta_{i,\xi}, 0),\\ \nabla v_{n,\xi} &\equiv (\nabla v^{+}_{n,\xi}, \nabla v^{-}_{n,\xi}) = (0, e^{(M-n)\partial}(-1)^{i}\delta_{i,\xi}).\end{aligned}$$

In such a way one can obtain the following expressions:

$$\begin{aligned}\{u_{n,i}, u_{m,j}\}_1 &= (-1)^{j}(\delta^{-}_{n,K} + \delta^{-}_{m,K} - 1)\\ &\quad (u_{n+m-K,i}\delta_{i,j+n-K} - (-1)^{(m+K)(n+K+1)}u_{n+m-K,j}\delta_{i,j-m+K}),\\ \{u_{n,i}, v_{m,j}\}_1 &= (-1)^{j}\Big[(\delta^{-}_{n,K} - 1)(v_{m-n+K,i}\delta_{i,j+n-K} - (-1)^{(m+M)(n+K+1)}v_{m-n+K,j}\delta_{i,j+m-M})\\ &\quad -\delta^{+}_{m,M}(u_{n-m+M,i}\delta_{i,j+n-K} - (-1)^{(m+M)(n+K+1)}u_{n-m+M,j}\delta_{i,j+m-M})\Big],\\ \{v_{n,i}, v_{m,j}\}_1 &= (-1)^{j}(1 - \delta^{+}_{n,M} - \delta^{+}_{m,M})\\ &\quad (v_{n+m-M,i}\delta_{i,j-n+M} - (-1)^{(m+M)(n+M+1)}u_{n+m-M,j}\delta_{i,j+m-M})\end{aligned} \tag{34}$$

for the first Hamiltonian structure and

$$\begin{aligned}&\{u_{n,i}, u_{m,j}\}_2 = -(-1)^{j}\frac{1}{2}\Big[u_{n,i}u_{m,j}(\delta_{i,j+n-K} - (-1)^{m}\delta_{i,j-m+K})\\ &\quad + \sum_{k=0}^{n+m}(\delta^{+}_{m,k} - \delta^{-}_{m,k})\Big((-1)^{mk}u_{n+m-k,i}u_{k,j}\delta_{i,j+n-k} - (-1)^{m(n+k+1)}u_{k,i}u_{n+m-k,j}\delta_{i,j-m+k}\Big)\Big],\\ &\{u_{n,i}, v_{m,j}\}_2 = -(-1)^{j}\frac{1}{2}\Big[u_{n,i}v_{m,j}(\delta_{i,j} + \delta_{i,j+n-K} - (-1)^{m}(\delta_{i,j+m-M} + \delta_{i,j+n+m-K-M}))\\ &\quad + 2\sum_{k=\max(0,m-n)}^{m-1}\Big(u_{n-m+k,i}v_{k,j+m-k}\delta_{i,j+n-K} - (-1)^{(n+1)m}v_{k,j}u_{n-m+k,i+k-m}\delta_{i,j+m+M}\Big)\Big],\\ &\{v_{n,i}, v_{m,j}\}_2 = (-1)^{j}\frac{1}{2}\Big[v_{n,i}v_{m,j}(\delta_{i,j-n+M} - (-1)^{m}\delta_{i,j+m-M})\\ &\quad - \sum_{k=0}^{n+m}(\delta^{+}_{m,k} - \delta^{-}_{m,k})\Big((-1)^{mk}v_{n+m-k,i}v_{k,j}\delta_{i,j-n+k} - (-1)^{m(n+k+1)}v_{k,i}v_{n+m-k,j}\delta_{i,j+m-k}\Big)\Big]\end{aligned}$$

for the second Hamiltonian structure where

$$\delta^{+}_{n,m} = \begin{cases} 1, & \text{if} \quad n > m\\ 0, & \text{if} \quad n \le m\end{cases}, \qquad \delta^{-}_{n,m} = \begin{cases} 1, & \text{if} \quad n < m\\ 0, & \text{if} \quad n \ge m.\end{cases}$$

Let us remind that the second Hamiltonian structure is valid for even values of (K, M) only.

The Hamiltonian structures thus obtained possess the properties (13–15) with $d_{\mathbb{O}} = d_{L^{+}_{K}} = d_{L^{-}_{M}}$. Using them one can rewrite flows (26) for even values of (K, M) in the bi-Hamiltonian form

$$D^{\pm}_{s}\begin{pmatrix} u^{(r)}_{n,i}\\ v^{(r)}_{n,i}\end{pmatrix} = \{\begin{pmatrix} u^{(r)}_{n,i}\\ v^{(r)}_{n,i}\end{pmatrix}, H^{\pm}_{s+1}\}_1 = \{\begin{pmatrix} u^{(r)}_{n,i}\\ v^{(r)}_{n,i}\end{pmatrix}, H^{\pm}_{s}\}_2,$$

with Hamiltonians

$$H_s^+ = \frac{1}{s} str(L_K^+)_*^s = \frac{1}{s} \sum_{i=-\infty}^{\infty} (-1)^i u_{sK,i}^{(s)}, \qquad H_s^- = \frac{1}{s} str(L_M^-)_*^s = \frac{1}{s} \sum_{i=-\infty}^{\infty} (-1)^i v_{sM,i}^{(s)}. \tag{35}$$

For odd values of (K, M) one can reproduce the bosonic flows of (26) only. In this case eqs. (35), due to relation (18), give only fermionic nonzero Hamiltonians using which the bosonic flows can be generated via odd first Hamiltonian structure (34)

$$D_{2s}^{\pm} \left(\begin{array}{c} u_{n,i}^{(r)} \\ v_{n,i}^{(r)} \end{array} \right) = \{ \left(\begin{array}{c} u_{n,i}^{(r)} \\ v_{n,i}^{(r)} \end{array} \right), H_{2s+1}^{\pm} \}_1.$$

6 Conclusion

In this paper, we have generalized the R-matrix method to the case of Z_2-graded operators with an involution and found that there exist two Poisson bracket structures. The first Poisson bracket is defined for both odd and even operators with Z_2-grading while the second one is found for even operators only. It was shown that properties of the Poisson brackets were provided by the properties of the generalized graded bracket. Then we have proposed the Lax-pair representation in terms of the generalized graded bracket of the new 2D fermionic (K, M)-Toda lattice hierarchy and applied the developed R-matrix formalism to derive its bi-Hamiltonian srtucture. For even values of (K, M) both even first and second Hamiltonian structures were obtained and for this case all the flows of the 2D fermionic (K, M)-TL hierarchy can be rewritten in a bi-Hamiltonian form. For odd values of (K, M) odd first Hamiltonian structure was found and for this case only bosonic flows of the 2D fermionic (K, M)-TL hierarchy can be represented in a Hamiltonian form using fermionic Hamiltonians.

Thus, the problem of Hamiltonian description of the fermionic flows of the 2D fermionic (K, M)-TL hierarchy is still open. Other problems yet to be answered are the construction of the second Hamiltonian structure (if any) for odd Lax operators and of the Hamiltonian structures (if any) for Lax operators L_K^+ and L_M^- of opposite Z_2-parities. Last but not least is the question of interrelation between the graded modified Yang-Baxter equation (20) proposed in this paper and the graded classical Yang-Baxter equation introduced in the pioneer paper [20]. All these questions are a subject for future investigations.

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